A LIFE-CYCLE INTERGENERATIONAL MODEL CONSIDERING SOCIAL SECURITY*

Um Modelo de Ciclo de Vida Intergeracional Considerando Seguridade Social

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Abstract

The central objective of this paper is to deliver an intergenerational Life Cycle model considering the retirement process based on the pay-as-you-go system (PAYG). Our model is inspired by Baranzinis' approach, where the optimal consumption analysis is made in a two-class type and restricted by the capital variation of each one. Our main results observe the equilibrium solution of the consumption and capital stock to both classes and conclude that the PAYG system does interfere with the results, as well as the time preference to leave or not inheritance. The methodology approached here is the Pontryagins' Maximum Principle. We also applied a numerical simulation to ensure the robustness of our approach.

Keywords: Capital Accumulation; Life Cycle; PAYG; Retirement.
JEL Code: J26; D15; C61.

Resumo

O objetivo central deste trabalho é desenvolver um modelo de ciclo de vidas intergeracional, considerando aposentadorias baseadas no sistema pay-as-you-go (PAYG). Nosso modelo é inspirado no de Baranzini, que apresenta a análise do consumo ótimo em um sistema de duas classes e restrito a variação do capital em cada uma. Nossos principais resultados observam a solução ótima de equilíbrio do consumo e estoque de capital para ambas as classes, concluindo que o sistema PAYG interfere nos resultados, assim como, a taxa de preferência no tempo por deixar ou não herança. A metodologia utilizada aqui é a do Princípio do Máximo de Pontryage. Também foi aplicada uma simulação numérica para garantir a robustes do modelo.

Palavra-chave: Acumulação de Capital; Ciclo de Vida; PAYG, Aposentadoria.

Códigos JEL: J26; D15; C61.

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1. Introduction

> "Since the end of the nineteenth century significant progress towards the removal of very great disparities of wealth and income has been achieved through the instrument of direct taxation— income tax and surtax and death duties— especially in Great Britain."
>
> The General Theory, Keynes (1936, p.372)

The central objective of this paper is to develop an alternative approach to the Intergenerational Life Cycle model considering social security, inspired by the "pay as you go" system (PAYG). We deal with a theoretical analysis concerning when the retirement system guarantees class mobility, and if economic growth and wealth distribution are impacted by it. Here, we consider Baranzinis' model (1991, CH. 6), which deals with heterogeneous agents in a life cycle model and inheritance, which characterize his approach closer to the Keynesian line of thinking[1]. The subject of social security has become one of the main topics of economic and political discussion in recent years both in developed and uneven economies. According to Portella and de Souza (2021) instead of privatizing the Social Security system, it is necessary to solidify the one sustained by the government and its financing, to guarantee the principle of human dignity.

The principles of the Life Cycle theory were presented in the early 1950s by Modigliani and Brumberg (1954). They assume that individuals plan their behaviour of the level of consumption and savings, throughout their life cycle. The authors' objective was to explain the consumption patterns of individuals. Agents' savings decisions must consider the consumption and savings choices considering the total income during life and their retirement at some point, so individuals' consumption decisions impact aggregate production at the macroeconomic level. In response to the traditional approach, Balestra and Baranzini (1971) worked in a two-class growth model considering a heterogeneous agent. They raise some income distribution hypotheses, and their result shows, that in equilibrium, the rate of profit is equal to the natural rate of growth divided by the capitalists' propensity to save, which is the well-known "Cambridge Equation" developed by Pasinetti (1962).

Moving forward, Baranzini (1991) presented a theory of wealth distribution, considering life cycle and inheritance. His model established microeconomic foundations for the theory of growth and wealth distribution based on the post-Keynesian framework. Nell (2013) describes micro-foundations, as an optimization system considering the consumption

[1] For more information about the microfundations of Baranzini's model and the post-Keynesian perspective, see the "The Oxford Handbook of Post-Keynesian Economics Volume 1" in the Chapter named: "The Cambridge post-Keynesian School of Income and Wealth Distribution".

agents and rationality principles. For him, individuals plan their economic behaviour during their Life Cycle, as well as reconcile simultaneously their individual intergenerational decisions. According to Wolff (1988), the specification of a Life Cycle savings model with a two-class seems to be consistent with the Pasinettis' result regarding the rate of interest and productivity growth in steady-state equilibrium. Besides, in a steady-state, wealth inequality among individuals appears to remain constant over time. Such conclusions come to reinforce the validity of the introduction of the microeconomic foundations into the two-class fixed savings model.

Therefore, the model and their extensions [like Teixeira, Sugahara and Baranzini (2002); Baranzini, Bejuino and Teixeira (2003); Wei and Araujo (2009); Sugahara, Aragón, da Cunha, Perdigão (2016); and Góes and Teixeira (2022)] does not deal with the social security system, which interferes in the dynamic of the model, since the consumption of the retired persons is composed by the social security. In this vein, we decided to introduce the PAYG system in a balanced government budget, where the public revenue is used as income transfer from capitalists and younger workers to the retired workers' consumption. A similar approach was presented by Heijdra (2002) and Acemoglu (2009) where the workers and capitalists income tax support the pension of the retiree, so there is a transfer of resources from young people to the elderly in each period. Therefore, the income transfer mechanism (considering government activities), adjusts the amounts received and transferred without losing resources between actively workers and the retired. Hence, this is a system that shows a solution that suits the characteristics of the Life Cycle because it dealt with intertemporal activities.

Our extension is based on the two-micro models' section treated in chapter 6 of Baranzinis' (1991) book and Góes and Teixeira (2022) where only capitalists are transmitting their wealth to their descendants, while workers accumulate only Life Cycle savings. These authors and we deal with the assumption that each individual makes his plans in order to maximize the value of the flow of discounted utilities from consumption, throughout his life expectancy. Besides, like them, we also deal with a continuous-time system.

For us, like Steedman (1972), the government agent uses taxation mechanisms to transfer income to the most vulnerable class (the workers), but in our case, is only to the retired workers. The taxation here will be direct on income as a mechanism for social security deducted from management, working as income transfer mechanisms. In our model, capitalists do not work and live off the consumption of the capital's income, then they do not retire. On the workers' side, we have that their taxation is only for future consumption in the retirement period. The redistribution of wealth is essential to the theory, first because wealth is a stock, and second because it is directly connected to economic growth. Keynes (1936, p. 372) affirmed that through direct taxation the state has achieved a lower concentration of disposable income and wealth. In this vein, we consider direct taxation on income, absorbed by the government agent, which will only transfer from one to another, and the public sector will not operate with deficits or surpluses.

This article will be divided into 7 sections: the first one is this introduction which presents the objectives and the justification of our new approach. The second one shows a historical overview of social security in theoretical models. After that, we present the principles and extensions of the Life Cycle approach. The fourth part introduces the settings of our model, presenting the structure and proposes. Section five presents the maximization, results, and comparative analysis. Section six presents the numerical simulation. Finally, section six presents the concluding remarks.

2. A Historical Overview of Social Security Modelling

With Harrods' (1948) discussion of "hump saving", the hypothesis that savings would be highest in the middle-age of a person's life as they saved for retirement, started to recognize the importance of saving during working years for consumption during retirement. According to Feldstein (1976), the process of Life Cycle saving has been radically altered by the growth of public programs of social security retirement benefits, moving forward, in 1978, the same author discusses how private pension programs differ from public social security.

Following Feldstein (1978), while public social security programs act as a substitute for family retirement savings, the private pension contributions made by an employer are a deductible business expense, and when the benefits are paid, they are considered taxable income for employees. Although such public social security programs are likely to reduce national savings, this tendency is offset by private pension programs. The private pensions could in principle decrease aggregate savings, if covered employees increase their consumption in the first period, more than in the comparison of the sum of pension-funded accumulation and the induced extra saving of shareholders. However, the growth of private pensions has not harmed savings. Indeed, it will have an increase in savings from a small amount caused by the combination of companies' partial funding, and the shareholders' response to unfunded liabilities.

Nevertheless, according to Gertler (1999), social security has positive results on capital intensity, such as real interest rates and labour supply. One implication is that social security will positively influence aggregate consumption, savings, and income distribution. However, there are two adjustments to a worker's wealth. The first one is when wealth includes the value of social security payments is equal to the one that workers can expect when he/she retires. The second is when the measure of human wealth is now net of a discounted tax stream. Although the increase in social security increases the capital stock, it has a strong negative wealth effect on the labour supply of retirees.

In the Brazilian case, the social security system is based on the parametric model, leading us to believe that this kind of system is not reliable for the near future. According to Holland and Malaga (2018), this system has a solidarity pillar between generations, thus, contributions financed by active young workers benefit the pensioners. This system is the parametric model and has been shown unsustainable thanks to population projections and changes in the labour market. Making it impossible to raise public incentives and reducing benefits for most Brazilians, turning the system inadequate. As a solution, the authors propose the adoption of a hybrid model, combining the advantages of the current repartition schemes with the capitalization regimen.

The first scheme is the PAYG method, which is used by the government as a theoretical model to guarantee social security. Here, the tax receipts are paid out as concurrent benefits and are not accumulated, when the current generation of workers retire, the benefits will depend on the tax payments of those who are active. Already, in the second scheme, each worker saves resources, which are saved in an individual self-account, which allows diversifying the individual risk among members of the same generation, this system has been in force in Chile since the 1981 reform. Thus, for Holland and Malaga (2018) the ideal structure would be to combine benefits paid by the government (universal) with a relatively low ceiling in distribution format, and more individual (private) contributions.

Following another line of thought, Miller (2020) states that none of the problems caused by public pensions is irreversible, public social security usually operates on a PAYG system. For him, the adoption of private pensions would not be a good alternative, since they are not redistributive, and do not guarantee lifetime benefits. Instead, it would increase inequality among seniors and would be less effective at reducing poverty than social security, therefore, it would not be a viable solution.

The goal is to narrow public services expenditure gaps by making ethical profits new markets based on collecting and trading debts are emerging. It is stated by Lavinas (2018) that financialization is making not only the poor increasingly dependent on credit and loans but also the middle classes. No more welfare states, but rather 'debt fare states', given that instead of consumer credit to buy commodities or services, people will consume debt. Being indebted and living in debt may become the norm, notably in times of neoliberal austerity policies, when the government seeks to cut spending.

Dreze and Khera (2017) analyse the case of India and conclude that the expansion of social security programs, together with the broader recognition of economic and social rights, have made an important contribution to human well-being. Therefore, Ellery Junior and Bugarin (2003) proved empirically that the social security system, PAYG, will contribute to the improvement of well-being, which presents a welfare gain compared to a fully saving-funded system.

In our model, we deal with an overlapping generation model that is useful for analysing social security [see Gertler (1999)]. In these models, individuals live two periods, one when they are young and active, and another when they are already inactive. In addition, we assume the existence of two classes, because, according to Balestra and Baranzini (1971), the ideal implications of a model with heterogeneous agents concerning the economy present conveniently synthesized results. These results are valid when you have a system that presents the balanced profit rate with balanced growth equal to the natural growth rate divided by the capitalists' propensity to save, known as the "Cambridge Equation" (Pasinetti, 1962). Second, the equilibrium profit-share, which shows how income is distributed, is also independent of the workers' propensity to save. Again, the worker's saving behaviour is irrelevant for determining results, that is, they are not strong enough to impact the system.

Retirement behaves as a PAYG system, and takes place during the life cycle, which is done by the government. The government must intervene in order to avoid situations that would become socially unbearable, as stated by Pasinetti (2012), either through government spending or through the redistribution of income. Here, this redistribution will be done via imposter charges, for the most vulnerable class, the workers, as well as in Steedman (1972). Foley and Michal (1999, p. 225) say: "Government taxes and transfers can have effects on the allocation of resources if the taxes and transfers are linked to economic decision variables like saving or profit". In the next section, we show a historical overview of the cycle theory.

3. Principles of the cycle theory and extensions

In the early 1950s, Franco Modigliani, among others, developed a theory assuming that individuals plan their behaviour of the level of consumption and savings, throughout their Life Cycle. Following this line, Baranzini (1991) developed a theory of wealth distribution, considering intergenerational theory and reason for inheritance. His model establishes microeconomic foundations for the theory of growth and distribution based on the post-Keynesian framework. According to Blecker and Setterfield (2019, p. 9), in post-Keynesian models, growth is fundamentally a demand-led process, as in Baranzinis' (1991) model. In this way, even the seemingly supply-determined limits to economic activity at any point in time are, in fact, likely to be influenced by the demand side of the economy. Baranzinis' objective would be to justify a better allocation of resources and economic growth. Thus, it is possible to optimize the utility of the agents and verify their behaviour over the period. In his model, the economy is divided between capitalists and workers, where the first-class receives profits and inheritance, while the second receives salaries. Several extensions of his approach were proposed.

Teixeira, Sugahara and Baranzini (2002) introduced government activities to the model, considering inheritance tax only. In their paper, they have shown how government transfers may be supported by orthodox micro-foundations. They expanded Baranzini (1991) in a discrete-time model dealing with capital accumulation, income distribution and inter-generational bequests. They analysed an approach only considering two periods, and in their case, there only exists taxation on inheritance. In the work, they conclude that the inclusion of the transfers' assumption is relatively negative to the capitalists' share in the total capital stock. Therefore, the Life Cycle hypothesis and bequest motive are compatible with basic governmental activities within a post-Keynesian framework. Baranzini, Benjuino, and Teixeira (2003) including taxation on capitalists' intergenerational bequests, also assume that such levy is fully transferred to the workers, who do not leave bequests to their descendants. Within this work, it is possible to show that total capital, as well as total savings, will expand. The later paper obtains these results in a continuous-time model.

Wei and Araujo (2009) provided an analysis of the optimal taxation to capitalists and workers in the Baranzinis' approach, intending to understand the public finances behaviour.

Their analysis spread the leisure and activity of the workers class, intending to verify in which situation they will choose to do work or not. Another accomplishment made by them is that, in the long run, the capital taxation tends to be zero, and in this case, the model is led to verify the equalization between profit rate and intertemporal preference tax. According to them, the workers can supply more or fewer jobs depending on the wages tax rate. In conclusion, their analysis shows that worker's taxation does not influence capital accumulation.

Sugahara, Aragón, da Cunha, Perdigão (2016) pursued to conceive orthodox micro-foundations to the macroeconomic model with the government. For this purpose, they used the overlapping generations with heterogeneous agents and the government model to allow both classes (capitalist and worker) to keep a positive intergenerational stock. Besides, they incorporated representative agents in heterogeneous classes. The main result of their work was the taxation effects on wealth distribution between classes. They also conclude a positive effect on the interest rates since the government presence increases the participation of the working class in the total capital stock of the economy. It is worth noting whether the provision of workers class in leaving an inheritance to their descendants is very high, they can disappearance with the capitalist class, leading to the euthanasia of the capitalist.

Another extension was designed by Góes and Teixeira (2022) when they show an alternative approach to the Baranzini (1991) model, presented for the continuous-time case. This paper is allowed technical progress and introduced behavioural differences between renters and workers. It is not only from the point of view of their initial endowments since in their case both classes can leave and receive the inheritance, depending on their preferences. They concluded that it has a possibility for workers to leave an inheritance and, in this case, does not change their functional distribution of income. However, it does change their consumption and capital stock (evidence of their bequest motive), expanding the growth of the economy. Therefore, this theory did not consider how the social security system can impact the system's results, which is one of our objectives.

4. Settings of the Model[2]

As stated in the introduction, the distinctive feature of the present model is the assumption of a differentiated interest rate. According to Balestra and Baranzini (1971),

[2] The Notations are in Appendix 1.

several reasons may be adduced in support of this assumption; first, historically, the interest rate has been considerably lower than the average profit rate; second, one can argue that the act of saving and the act of investing are two distinct operations; third, there is a risk factor associated with the investment, this risk should be reflected in the profit rate; fourth, it may be said that investment, to be profitable, must be carried out in a certain minimum quantity, the workers, taken individually, are not able to exploit the profit opportunities of big investment. Besides that, we shall assume a well-behaved neoclassical production function $Y = f(K, L)$, with constant returns to scale and possesses positive first-order partial derivatives and negative second-order direct partial derivatives.

A key feature of the neoclassical growth model is that they admit a representative household for the analysis of capital accumulation. Moreover, allow us to establish the equivalence between equilibrium and optimization growth problems. However, this assumption is not appropriate as the population grows, which the case is presented here. This occurs because, in particular, decisions made by the younger generation will affect the elderly. These economic interactions have no counterpart in the neoclassical growth model. Acemoglu (2009) affirms that these models are useful for several reasons. First, they capture the potential interaction of different generations of individuals in the marketplace. Second, they provide a tractable alternative to the infinite-horizon representative agent models. Third, some of their key implications are different from those based on the neoclassical growth model, as well as the dynamics of capital accumulation and consumption. The simplest way to relax the representative households' assumption is accomplished by the introduction of two-period lives.

To obtain explicit analytical solutions, we consider a two-period model, based on the Baranzini (1991, CH 6) approach. Individuals are fully trained adults at $t = 0$ when they start earning income, their activity period is for R years and then they retire. Each one dies at age T, so enjoying $T - R$ years of retirement. This allows us to observe the effects of the various parameters like the savings rates, taxation, and capital accumulation in a more generalized Life Cycle model of income and wealth distribution. For both individuals the criterion of choice maximizes the value of the discount utilities, as individuals seek to maximize their consumption, these are:

$$U_C = \int_0^T e^{-\delta_C t} U_C[c_C(t)]dt = \int_0^T e^{-\delta_C t} \frac{1}{a} [c_C(t)]^a dt \tag{1}$$

$$U_W^A = \int_0^R e^{-\delta wt} U_W^A[c_W^A(t)]dt = \int_0^R e^{-\delta wt} \frac{1}{a} [c_W^A(t)]^a dt \tag{2}$$

$$U_W^R = \int_R^T e^{\eta t} U_W^R[c_W^R(t)]dt = \int_R^T e^{(1+\eta)t} \frac{1}{a} [c_W^R(t)]^a dt \tag{3}$$

Like in Baranzini (1991, p. 158) we are using the CRRA utility function. Equation (1) and (2) shows the general utility functions of the capitalist and activity workers, which are negatively affected by the intertemporal pure-time preference of each class. Equation (3) is the utility of the retired workers and, since these workers does not leave inheritance, they are not impacted by the intertemporal pure-time preference. Therefore, since the PAYG represents the income transfer from capitalists and activities workers to the retired workers, the population growth rate affects the level of their consumption. Each class has their consumptions rates and are defined as:

$$c_c(t) = c_c(0)e^{g_{c_c}^* t} = k_0 \left(r - g_{c_c}^*\right) \frac{1 - e^{R(n-r)}}{1 - e^{T(g^*-r)}} e^{g_{c_c}^* t} \tag{4}$$

$$c_W^A(t) = c_W^A(0)e^{g_{c_W^A} t} \tag{5}$$

$$c_W^R(t) = c_W^R(R)e^{g_{c_W^R} t} \tag{6}$$

where: $g_{c_c}^* = \frac{r-\delta}{1-a}$; $g_{c_W^A} = \frac{i-\delta_W}{1-a}$; $g_{c_W^R} = \frac{\eta+i}{1-a}$.[3]

Equations (4), (5), and (6) represent the consumption of the capitalists, actively and retired workers respectively. The similarity between us, Baranzini (1991) and Góes and Teixeira (2022) is that we all consider three consumption rates. The first one is for capitalists, the second for activity workers and the final is the retired workers. However, this model differs from Baranzini (1991) and Góes and Teixeira (2022) for considering imperfect markets, that is; $i < r$. Following Rust and Phelan (1997), the imperfection of markets is a necessary condition for Social Security to have a significant behavioural effect, because if individuals had access to a perfect market, they could design their own ideal retirement. Thus, the interest rate (r) is exogenous, capitalists have full access to perfect capital markets, and workers face a lower deposit interest rate (i) called the deposit interest rate, which only guarantees workers

[3] The mathematical algebraic manipulations of $g_{c_W^A}$ and $g_{c_W^R}$ are in Appendix 2A.

small savings to transfer income from the active period to the retired period. This assumption guarantees workers small savings to transfer income from working time to retirement time. Thus, it can be guaranteed that the contribution works as a social security system and not just as an income transfer process. In this vein, our model has three different growth rates of consumption: $g^*_{c_c}$, $g_{c^A_W}$,$g_{c^R_W}$.

$$c^R_W(t) = \left[(1+\eta)\left\{t_c r k_c(t)^* + t_W\left[w + i k^A_W(t)\right]\right\} + \overline{k^R_W}\right] \tag{7}$$

Equation (7) shows the income transferred from capitalists and activity workers to retired workers. This equation is the PAYG process definition, where the younger generation leaves a part of their incomes to the elderlies. This formula differs from the original model presented by Baranzini (1991), as well as, by their extensions like Teixeira, Sugahara and Baranzini (2002), Wei and Araujo (2009), Sugahara, Aragón, da Cunha, Perdigão (2016), and Góes and Teixeira (2022). This Equation guarantees the minimum of survives, since, even exogenously, the government can impose a minimum value for each taxation, to ensure at least the same level of the consumption of the activity workers. In this case, we have a guarantee that the class mobility sustains, and the retired workers do not deteriorate their consumption. The maximization of its utility is subjective to its restriction of the capital stock variation, thus:

$$\dot{k}_C = (1 - t_C) r k_C - c_C \tag{8}$$

$$\dot{k}^A_W = (1 - t_W)\left(w + i k^A_W\right) - c^A_W \tag{9}$$

$$\dot{k}^R_W = (1+\eta)\left\{t_c r k_c + t_W\left[w + i k^A_W(t)\right]\right\} + \overline{k^R_W} - c^R_W(t) \tag{10}$$

Formulas (8) and (9) defines the public sector revenue by taxing the incomes and guarantee the workers personal income distribution in R. These equations show the behaviour of each capital stock (savings/investment functions). The taxations dealt here are direct on income, in addition, it works as a mechanism for social security discounting the inheritance. Equation (10) also represents the behaviour of the capital to the retired, however, since they do not save, this equation is null. In this vein, they will consume all their income provenience from the PAYG and a fixed amount invested when they were activity workers. Note that, since the workforce is raising or decreasing in time, the income provenience from PAYG depends on the level of the population growth rate.

As proposed by Kaldor (1955-6), the tax would be imposed as a personal tax at the level of the households, with progressive rates applicable to aggregate consumption. This allows the financing of social security, transferring part of the income collected by the capital accumulation to the most vulnerable class, the workers. The taxation of capitalists and younger workers deals as mechanisms for transferring income to the retirement of workers. For Hicks (1999) social insurance programs are more than expedient outcomes of particular working-class encounters with other powers. So, these are rights of the working class, which makes only that class retire, since capitalists do not work and live off the consumption of the capital's income, then they do not retire. From these equations, we have our system to analyse the effects of retirement on the model. This analysis is approached in the next section by using the Pontryagin Maximum Principle.

5. Maximization and optimal control

Optimal control is a set of differential equations describing the paths of the control variables that minimizes or maximizes the objective function. The optimal control problem takes into account the objective function, what we want to maximize, the equations that model the dynamics of the problem at each instant of time and the constraints of the problem. Finally, the initial and final conditions, intend to find, in a set of possible solutions, the admissible set that maximizes (minimizes) the objective function. The most important result in the theory of optimal control is the "Maximum Principle" (developed by Pontryangin, 1987), which is a fundamental result to obtaining the optimal solution to an optimal control problem. These are necessary optimality conditions, through this result, we have one method of finding candidates for optimal controls is by the constructive utilization of necessary conditions for optimality such in dynamic optimization problems.

In economics, we deal with the rationality principle, first presented by Turgot (1793), and structured as a microeconomic axiom by Walras (1874). He presented that all based agents in economies (for us, capitalists, and workers) look to maximize their utilities. According to Dorfman (1969), the basic equations of the maximum principle are the limit forms of the first-order necessary conditions for a maximum applied to the same problem. Besides that, he affirms "the same results deduced from the more familiar method of maximizing subject to a finite number of constraints." Dorfman (1969, p.827). Such an approach used here is shown in chapter 4 by Léonard and Van Long (1992), which follows the above orders:

1st - We have to determine our objective function and its restrictions.

2nd - We have to construct the Hamiltonian which will be optimized.

3rd - We have to deliver each first-order condition between the Hamiltonian and the state and costate variables.

4th - We have to approach the maximal principle.

Proposition 1 - The PAYG retirement process depends on the amount of the capitalists' and workers' incomes, population growth rate and the growth rate of the consumption since their consumption level must be determined. The government revenue came from direct taxation on their incomes, which is transferred to the retired workers. Thus, since the retired workers do not leave an inheritance, they will consume all their income transferred and the capital accumulated when they were active workers. In this vein, considering the systems ($\dot{c}_C$,$\dot{k}_C$), ($\dot{c}_W^A$,$\dot{k}_W^A$), and ($\dot{c}_W^R$,$\dot{k}_W^R$). In the dynamic formulas, we have to define the optimal solution for each class.[4]

Proof: Our problem sustains in a maximization problem, in which we must maximize the following utility functions:

$$\text{Max}U_C = \int_0^T e^{-\delta_C t}\frac{1}{a}[c_C(t)]^a dt$$

$$\text{Max}U_W^A = \int_0^R e^{-\delta_W t}\frac{1}{a}\left[c_W^A(t)\right]^a dt$$

$$\text{Max}U_W^R = \int_R^T e^{(1+\eta)t}\frac{1}{a}\left[c_W^R(t)\right]^a dt$$

Their restrictions are:

$$\dot{k}_C = (1 - t_C)rk_C(t) - c_C(t)$$

$$\dot{k}_W^A = (1 - t_W)\left[w + ik_W^A(t)\right] - c_W^A(t)$$

$$\dot{k}_W^R = (1 + \eta)\left\{t_c rk_c + t_W\left[w + ik_W^A(t)\right]\right\} + \overline{k_W^R} - c_W^R(t)$$

In our model, the consumption functions are (4), (5), and (6), which determines the utility function (objective function), and the restrictions are (8), (9), and (10). Thus, we can structure our capitalists and workers Hamiltonians respectively, as:

$$H_c = e^{-\delta_C t}\frac{1}{a}[c(t)]^a + \lambda_1[(1 - t_C)rk_C(t) - c_C(t)]$$

[4] The Appendix 2A and 2B contains the mathematical proofs.

$$H_W^A = e^{-\delta w t}\frac{1}{a}\left[c_W^A(t)\right]^a + e^{nt}\frac{1}{a}\left[c_W^R(t)\right]^a + \lambda_2\left[(1-t_W)\left(w + ik_W^A(t)\right) - c_W^A(t)\right]$$

$$H_W^R = e^{(1+\eta)t}\frac{1}{a}\left[c_W^R(t)\right]^a + \lambda_3\left[(1+\eta)\{t_c r k_c(t) + t_W[w + ik_W^A(t)]\} + k_W^{\acute{R}} - c_W^R(t)\right]$$

Approaching the first-order condition for each Hamiltonian, we have the following results:

$$\frac{\partial H_C}{\partial c_C(t)} = c_C(t)^{a-1}e^{-\delta_c t} - \lambda_1 = 0 \rightarrow \lambda_1 = c_C(t)^{a-1}e^{-\delta_c t}$$

$$\dot{\lambda}_1(t) = \frac{-\partial H_C}{\partial k_C(t)} = -\lambda_1(1-t_C)r$$

$$\dot{k}_C(t) = \frac{\partial H_C}{\partial \lambda_1(t)} = (1-t_C)rk_C(t) - c_C(t)$$

$$\frac{\partial H_W^A}{\partial c_W^A(t)} = 0 = e^{-\delta w t}c_W^A(t)^{a-1} - \lambda_2$$

$$\dot{\lambda}_2 = \frac{-\partial H_W^A}{\partial k_W^A(t)} = -(1-t_W)i$$

$$\dot{k}_W^A = \frac{\partial H_W^A}{\partial \lambda_2} = (1-t_W)(w + ik_W^A) - c_W^A(t)$$

$$\frac{\partial H_W^R}{\partial c_W^R(t)} = 0 = e^{(1+\eta)t}c_W^R(t)^{a-1} - \lambda_3$$

$$\dot{\lambda}_3 = \frac{-\partial H_W^R}{\partial k_W^R(t)} = 0$$

$$\dot{k}_W^R = 0 = \frac{\partial H_W^R}{\partial \lambda_3} = (1+\eta)\{t_c r k_c(t) + t_W[w + ik_W^A(t)]\} + k_W^{\acute{R}} - c_W^R(t)$$

Approaching the Principle of Pontryagin Maximum, we find the trajectory equations of the capital and consumption to each class. First, we analyse the capitalists' results.

$$k_C^*(t) = \left(\frac{1 - e^{R(n-r)}}{1 - e^{T(g_{c_C}^* - r)}}\right)\left(\frac{r - g_{c_C}^*}{r}\right)k_0 e^{g_{c_C}^* t} \tag{11}$$

$$c_C^*(t) = (1-t_C)\left(\frac{1 - e^{R(n-r)}}{1 - e^{T(g_{c_C}^* - r)}}\right)(r - g_{c_C}^*)k_0 e^{g_{c_C}^* t} \tag{12}$$

Equation (11) depends on the level of the inheritance, but this result is not affected by the taxation. Our result differs from the one found by Góes and Teixeira (2022), in our case, the result grows exponentially in respect to the birth growth rate, and in their case is in respect also to the profit ratio. Equation (12) is directly and negatively affected by the taxation, transferring part of the consumption to the retired workers in this case, the only difference between us and Góes and Teixeira (2022) is the taxation. We must point out that the taxation initially only appears in the capital accumulation (the restriction) formula, however, in equilibrium condition we actually have a compression of the capitalists' consumption $\left[\frac{\partial c_C^*(t)}{\partial t_C} < 0\right]$ to guarantee retirement. Analytically speaking, the income transfer came from consumption and not from the total capital. Therefore, since the capital stock is rising, this does not mean a prejudice against capitalists. Now, the workers' analysis:

$$k_W^{A\,*} = \left[\frac{\delta_W e^{-\delta_W t}}{[(1-t_W)i]^a}\right]^{\frac{1}{1-a}} - \frac{w_o e^{mt}}{i} \quad (13)$$

$$c_W^A(t)^* = \left[\frac{(1-t_W)i}{\delta_W e^{-\delta_W t}}\right]^{\frac{1}{a-1}} = \left[\frac{\delta_W e^{-\delta_W t}}{(1-t_W)i}\right]^{\frac{1}{1-a}} \quad (14)$$

Equation (13) is the stock of the activity worker's and, differently from the capitalists, now the taxation affects the amount, Wei and Araujo (2009) conclude that wages taxation does not affect the capital accumulation, however, in our model does $\left[\frac{\partial k_W^{A\,*}}{\partial t_W} > 0\right]$. Our result differs from the one found by Góes and Teixeira (2022) showing the taxation impact which restructured all the formulas. The same difference is presented in (10), which shows a positive relationship between the worker's taxation and the consumption $\left[\frac{\partial c_W^A(t)^*}{\partial t_W} > 0\right]$, reflecting the increase of retired consumption and the activity workers' capital stock. We also find the optimal consumption of the retired workers.

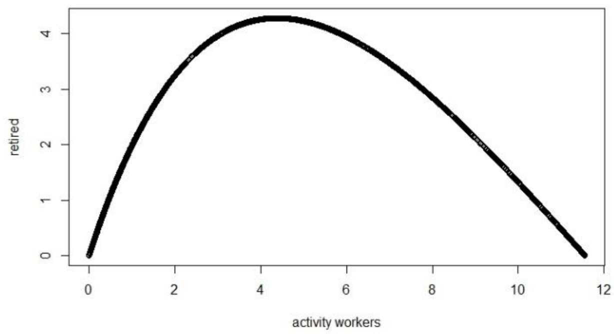
$$c_W^R(t)^* = (1+\eta)\{t_c r k_c(t)^* + t_W[w + i k_W^A(t)^*]\} + \overline{k_W^R} \quad (15)$$

Equation (15) shows the retired workers' consumption in equilibrium. It is interesting the growth rate of the population affects this equilibrium, and that conclusion works to present the robustness of the PAYG system. This result agrees to the one presented in Nogueira Silva, Morreto and Kappes (2021) conclusions. We also accomplish this by considering that workers' do save part of their income when they are activity workers, and the amount saved will be consumed when they are retired. The next section presents the numerical simulation.

6. Numerical Simulation and Analysis

According to Lavoie (2014, p. 37), computational analysis, providing more ammunition in the unorthodox search for an explanation and causal mechanisms, is a powerful weapon in the battle of ideas. Appendix 3 presents the table with the specific variables and values. This section approached the Dynamic Ordinary Equation computational analysis, by using the Software R with the deSolve path. Our interest here is to provide a numerical simulation to justify the behaviour of our model. In this section, we properly analyse the capital accumulation: capitalists, retired workers and activity workers. Figure 1 simulates the capital accumulation of the worker's class:

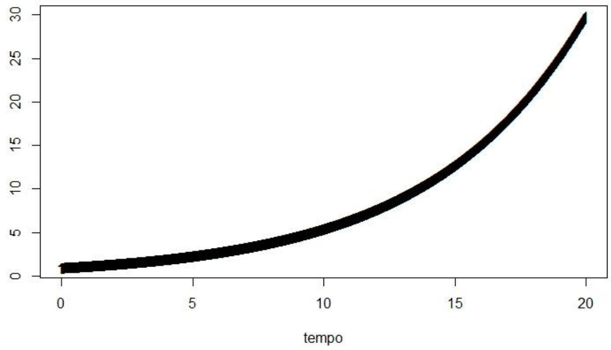
Figure 1 - Worker's capital accumulation



Source: authors

As we can see, Figure 1 proves that, along the worker's life, the active ones will accumulate their capital, but the retired ones do not leave inheritance and, at first, they will accumulate some capital, since they are receiving capital from government PAYG process, but at some point they will increase their consumption faster than the accumulation and will consume all the capital up to death. This assumption is plausible for real life, since old people needs more resources to guarantee the minimum for subsistence, since they will need to spend more with medicine, health food, exercises, insurance and other additional costs. Figure 2 shows the capitalist capital accumulation, thus:

Figure 2 - Capitalist Capital Accumulation



Source: authors

The second figure shows that at first the capitalists only have one piece of capital, but along the time they will accumulate capital and, since they leave inheritance, the final point is the amount of capital leaved to the next generation of capitalists. It is interesting that the government activity exponentiated the capitalist capital accumulation, which confirms the effects on the profit rate presented by Steedman (1972) and discussed in this paper. The consumption of the capitalists is really small in comparison to their earns, and this guarantee the transfer of capital between generations. The next section presents the concluding remarks.

7. Concluding Remarks

Our paper contributes to four aspects:

(I) We modelled Baranzinis' model concerning an imperfect markets and new consumption function of the capitalists, activity, and retired workers. We consider direct taxation on the capitalists and activity workers income. Here, we show three different consumption growth rates since the capitalists and workers will have different interest rates. In this vein, with these new assumptions, we structured a system that made possible the analysis of the personal income distribution behaviour.

(II) Considering the new formulation, we could approach the Principle of Pontryagin Maximum, to deliver the optimal capital stock and consumption to each class. We proved the impact of the populational growth rate to retired consumption, which ensures the robustness of the PAYG system.

(III) We construct, from the beginning to the end of this article a historical and theoretical debate around the pension theme.

(IV) The numerical simulation confirms the results presented in section 5, especially the Steedman's Cambridge Equation proposal.

To conclude, the results obtained in the present article advance the economic literature. However, there remain considerable analytical issues to deal with, like an empirical analysis or the stability condition of the model.

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APPENDIX 1: NOTATIONS

$\dot{c}_C$	Variation of capitalists' consumption on time
$\dot{c}_W$	Variation of workers' consumption on time
$\dot{k}_C$	Variation of capitalists' capital stock on time
$\dot{k}_C^a$	Variation of the capital stock owned by capitalist who is active
$\dot{k}_C^p$	Variation of the retired capitalists' stock of capital on time
$\dot{k}_W$	Variation of workers' capital stock on time
$\dot{k}_W^a$	Variation of the active workers' stock of capital on time
$\dot{k}_W^p$	Variation of the retired workers' stock of capital on time
$c_c(t)$	Capitalists' consumption at time t
$c_w(t)$	Workers' consumption at time t
c_w^A	Workers' consumption when they are active
c_w^R	Workers' consumption when they are retired
e^{rk}	Consumption at time $t + k$
g^*	Number of births rate in optimal
H_C	Capitalists' Hamiltonian
H_W	Workers' Hamiltonian
k_0	Capital stock that they have inherited.
k_C	Capitalists' capital stock
k_c^a	Capital stock owned by a capitalist during who is active

k_c^p	The financial capital stock of a retired capitalist
k_w	Workers' capital stock
k_w^a	Capital stock owned by a worker during who is active
k_w^p	Capital stock owned by a worker during who is retired
t_c	Capital taxation
t_w	Wage taxation
U_c	Utility function for consumption from the capitalists
U_w	Utility function for consumption from the workers
δ_c	Capitalists' Pure time-preference
δ_w	Workers' Pure time-preference
λ_i	Shadow price, $i = 1,2$
B	Bequest
$c(t)$	Consumption at time t
H	Hamiltonian
i	Deposit interest rate
K	Capital
L	Labour
m	Rate of labour-augmenting technical progress
n	The natural rate of growth
p	A fixed proportion of the wage-rate he would earn if he was still working

R	Retirement age
r	Rate of interest
T	Die age
t	Borne age
v	End of the active period
w	Wage-rate
Y	Output
η	Population growth rate
a	time-preference
g	Number of births rate

APPENDIX 2 – MATHEMATICAL MANIPULATIONS

A - CAPITALISTS

$\mathrm{Max}U_C = \int_0^T e^{-\delta t}\frac{1}{a}[c(t)]^a dt$

s.t. $\dot{k}_C = (1 - t_C)rk_C - c_C$

The Hamiltonian:

$H_c = e^{-\delta t}\frac{1}{a}[c(t)]^a + \lambda_1[(1 - t_C)rk_C - c_C(t)]$

The first-order conditions:

$$\frac{\partial H_C}{\partial c_C(t)} = c_C(t)^{a-1}e^{-\delta t} - \lambda_1 = 0 \rightarrow \lambda_1 = c_C(t)^{a-1}e^{-\delta t} \qquad (1A)$$

$$\dot{\lambda}_1(t) = \frac{-\partial H_C}{\partial k_C(t)} = -\lambda_1(1 - t_C)r \qquad (2A)$$

$$\dot{k}_C(t) = \frac{\partial H_C}{\partial \lambda_1(t)} = (1 - t_C)rk_C - c_C(t) \qquad (3A)$$

From (1), we find the second $\dot{\lambda}_1(t)$:

$$\dot{\lambda}_1 = (a - 1)c_C(t)^{a-2}\dot{c}_C e^{-\delta_C t} - \delta e^{-\delta_C t}c_C(t)^{a-1} \qquad (4A)$$

Equalizing (4) and (2)

$$-c_C(t)^{a-1}e^{-\delta_C t}(1 - t_C)r + \delta_C e^{-\delta_C t}c_C(t)^{a-1} = (a - 1)c_C(t)^{a-2}\dot{c}_C e^{-\delta_C t}$$

$$\frac{c_C(t)[\delta_C - (1 - t_C)r]}{(a-1)} = \dot{c}_C \qquad (5A)$$

From (5), we have t_C^*:

$$\frac{c_C(t)[\delta_C - (1 - t_C)r]}{(a-1)} = 0 \rightarrow \delta_C - (1 - t_C)r = 0 \rightarrow (1 - t_C) = \frac{\delta_C}{r} \rightarrow t_C^* = 1 - \frac{\delta_C}{r} \qquad (6A)$$

From (3) and (6), we have $c_C(t)$

$$c_C(t) = \left(1 - 1 + \frac{\delta_C}{r}\right) rk_C \rightarrow c_C^*(t) = \delta_C k_C^*(t) \tag{7A}$$

We have capitalists' capital stock, from Baranzini (1991) and Góes and Teixeira (2020):

$$k_c^a(t) = k_0 e^{nt} + \int_0^t [(r-n)k_0 e^{nt} - c_c(v)] e^{r(t-v)} dv = k_0 e^{rt}\left[1 - B\left(1 - e^{(g^*-r)t}\right)\right] \tag{8A}$$

$$k_c^p(t) = [k|c^a(R) - k_0 e^{Rn}] e^{r(t-R)} + \int_R^t c_c(v) e^{r(t-v)} dv = k_0 e^{rt}\left(e^{(g^*-r)t} - e^{(g^*-r)T}\right) \tag{9A}$$

Thus:

$$k_C(t) = k_C^a + k_C^p \rightarrow \ln[k_C(t)] = \ln\left[k_C^a + k_C^p\right] \rightarrow \frac{\dot{k}_C}{k_C} = \frac{\dot{k}_C^a}{k_C} + \frac{\dot{k}_C^p}{k_C} \tag{10A}$$

From (8), we have:

$$k_C^a = k_0 e^{rt}\left[1 - B\left(1 - e^{(g^*-r)t}\right)\right] \rightarrow \ln(k_C^a) = \ln k_0 + rt + \ln\left[1 - B\left(1 - e^{(g^*-r)t}\right)\right]$$

$$\frac{\dot{k}_C^a}{k_C^a} = r + \frac{(g^*-r)Be^{(g^*-r)t}k_0 e^{rt}}{k_c^a} \rightarrow \dot{k}_C^a = rk_c^a + (g^* - r)Be^{(g^*-r)t}k_0 e^{rt} \tag{11A}$$

Considering $\dot{k}_C^a = 0$, thus:

$$k_C^a(t)^* = \frac{(g^*-r)Be^{(g^*-r)t}k_0 e^{rt}}{r} \tag{12A}$$

From (9), we have:

$$\frac{\dot{k}_c^p}{k_c^p} = r + \frac{(g^*-r)\left[e^{(g^*-r)t} - e^{(g^*-r)T}\right]}{e^{(g^*-r)t} - e^{(g^*-r)T}} \rightarrow \frac{\dot{k}_c^p}{k_c^p} = r + (g^* - r) \rightarrow \dot{k}_c^p = g^* k_c^p \tag{13A}$$

From (10), we find:

$$\frac{\dot{k}_C}{k_C} = \frac{\dot{k}_C^a}{k_C} + \frac{\dot{k}_C^p}{k_C} \rightarrow \dot{k}_C = \dot{k}_C^a + \dot{k}_C^p \rightarrow \dot{k}_C = rk_C^a + (g^* - r)Be^{(g^*-r)t}k_0e^{rt} + g^*k_c^p$$

$$k_C(t)^* = Be^{(g^*-r)t}k_0e^{rt}\frac{(r-g^*)}{r} \tag{14A}$$

Since: $B = \left(\frac{1-e^{R(n-r)}}{1-e^{T(g^*-r)}}\right)$, see Góes and Teixeira (2022), thus, we have the optimal points:

$$k_C(t)^* = \left(\frac{1-e^{R(n-r)}}{1-e^{T(g^*-r)}}\right)e^{(g^*-r)t}k_0e^{rt}\frac{(r-g^*)}{r} \tag{15A}$$

$$c_C^*(t) = \delta_C\left(\frac{1-e^{R(n-r)}}{1-e^{T(g^*-r)}}\right)e^{(g^*-r)t}k_0e^{rt}\frac{(r-g^*)}{r}$$

Since: $t_C^* = 1 - \frac{\delta_C}{r}$, thus:

$$c_C^*(t) = (1 - t_C)\left(\frac{1-e^{R(n-r)}}{1-e^{T(g^*-r)}}\right)e^{(g^*-r)t}k_0e^{rt}(r - g^*) \tag{16A}$$

B - WORKERS

1st – Activity Workers

The Hamiltonian:

$$H_W^A = e^{-\delta wt}\frac{1}{a}\left[c_W^A(t)\right]^a + \lambda_2\left[(1 - t_W)\left(w + ik_W^A\right) - c_W^A(t)\right] \tag{1C}$$

Applying the first-order conditions:

$$\frac{\partial H_W^A}{\partial c_W^A(t)} = 0 = e^{-\delta wt}c_W^A(t)^{a-1} - \lambda_2 \tag{2C}$$

$$\dot{\lambda}_2 = \frac{-\partial H_W^A}{\partial k_W^A(t)} = -(1 - t_W)i \tag{3C}$$

$$\dot{k}_W^A = \frac{\partial H_W^A}{\partial \lambda_2} = (1 - t_W)\left(w + ik_W^A\right) - c_W^A(t) \tag{4C}$$

Deriving (2C) with respect to time:

$$\dot{\lambda}_2 = -\delta_W e^{-\delta wt}c_W^A(t)^{a-1} + e^{-\delta wt}(a - 1)c_W^A(t)^{a-2}\dot{c}_W^A \tag{5C}$$

Equalizing (5C) and (3C):

$$-(1-t_W)i = -\delta_W e^{-\delta_W t} c_W^A(t)^{a-1} + e^{-\delta_W t}(a-1)c_W^A(t)^{a-2}\dot{c}_W^A \tag{6C}$$

Isolating $\dot{c}_W^A$ in (6C):

$$\frac{[\delta_W e^{-\delta_W t} c_W^A(t)^{a-1} - (1-t_W)i]}{e^{-\delta_W t}(a-1)c_W^A(t)^{a-2}} = \dot{c}_W^A \tag{7C}$$

Considering (7C), if $\dot{c}_W^A = 0$

$$\frac{[\delta_W e^{-\delta_W t} c_W^A(t)^{a-1} - (1-t_W)i]}{e^{-\delta_W t}(a-1)c_W^A(t)^{a-2}} = 0 \rightarrow c_W^A(t)^{a-1} = \frac{(1-t_W)i}{\delta_W e^{-\delta_W t}}$$

$$c_W^A(t)^* = \left[\frac{(1-t_W)i}{\delta_W e^{-\delta_W t}}\right]^{\frac{1}{a-1}} = \left[\frac{\delta_W e^{-\delta_W t}}{(1-t_W)i}\right]^{\frac{1}{1-a}} \tag{8C}$$

From (4C), we have:

$$0 = (1-t_W)\left(w + ik_W^A\right) - c_W^A(t)^*$$

$$c_W^A(t)^* = (1-t_W)\left(w + ik_W^A\right)$$

$$c_W^A(t)^* - (1-t_W)w = (1-t_W)ik_W^A$$

$$k_W^{A\,*} = \frac{c_W^A(t)^*}{(1-t_W)i} - \frac{w}{i}$$

$$k_W^{A\,*} = \left[\frac{[(1-t_W)i]^a}{\delta_W e^{-\delta_W t}}\right]^{\frac{1}{a-1}} - \frac{w}{i} = \left[\frac{\delta_W e^{-\delta_W t}}{[(1-t_W)i]^a}\right]^{\frac{1}{1-a}} - \frac{w}{i}$$

Baranzini (1991, p. 163) define $w = w_o e^{mt}$.

$$k_W^{A\,*} = \left[\frac{\delta_W e^{-\delta_W t}}{[(1-t_W)i]^a}\right]^{\frac{1}{1-a}} - \frac{w_o e^{mt}}{i} \tag{9C}$$

2nd – Retired Workers:

The Hamiltonian:

$$H_W^R = e^{\eta t}\frac{1}{a}\left[c_W^R(t)\right]^a + \lambda_3\left[(1+\eta)\left\{t_c r k_c + t_W\left[w + ik_W^A(t)\right]\right\} + \overline{k_W^R} - c_W^R(t)\right] \quad (10C)$$

Approaching the first-order conditions:

$$\frac{\partial H_W^R}{\partial c_W^R(t)} = 0 = e^{\eta t}c_W^R(t)^{a-1} - \lambda_3 \quad (11C)$$

$$\dot{\lambda}_3 = \frac{-\partial H_W^R}{\partial k_W^R(t)} = 0 \quad (12C)$$

$$\dot{k}_W^R = 0 = \frac{\partial H_W^R}{\partial \lambda_3} = (1+\eta)\left\{t_c r k_c + t_W\left[w + ik_W^A(t)\right]\right\} + k_W^R - c_W^R(t) \quad (13C)$$

From (11C), we have:

$$0 = e^{\eta t}c_W^R(t)^{a-1} - \lambda_3 \rightarrow \lambda_3 = e^{\eta t}c_W^R(t)^{a-1}$$

$$\eta e^{\eta t}c_W^R(t)^{a-1} + e^{\eta t}(a-1)c_W^R(t)^{a-2}\dot{c}_W^R = 0$$

$$e^{\eta t}(a-1)c_W^R(t)^{a-2}\dot{c}_W^R = -\eta e^{\eta t}c_W^R(t)^{a-1}$$

$$\dot{c}_W^R = \frac{-\eta e^{\eta t}c_W^R(t)^{a-1}}{e^{\eta t}(a-1)c_W^R(t)^{a-2}}$$

$$\dot{c}_W^R = \frac{\eta e^{\eta t}c_W^R(t)^{a-1}}{e^{\eta t}(1-a)c_W^R(t)^{a-2}}$$

$$\dot{c}_W^R = \frac{\eta c_W^R(t)}{(1-a)} \quad (14C)$$

From (13C), we have:

$$0 = (1+\eta)\left\{t_c r k_c + t_W\left[w + ik_W^A(t)\right]\right\} + \overline{k_W^R} - c_W^R(t)$$

$$c_W^R(t)^* = (1+\eta)\left\{t_c r k_c(t)^* + t_W\left[w + ik_W^A(t)^*\right]\right\} + \overline{k_W^R} \quad (15C)$$

APPENDIX 3 – NUMERICAL SIMULATION VARIABLES AND VALUES

The values presented here are hypothetical, but based on estimated values for emerging economies such as Latin America countries. The initial values were calculated based on the parameters and/or theoretical interpretation.

Parameters

Parameter	Value
t_c	0,15
t_w	0,3
r	0,2
w	0,6
i	0,06
η	0,02
δ	0,14
δ_w	0,05
α	0,1
$k_w^{\acute{r}}$	1

Initial Values

Variable	Value
k_c	1
k_w^a	0
k_r^w	0
c_c	0,00036
c_w^a	0,04657
c_w^r	0,31267